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Test-field model for inhomogeneous turbulence

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The test-field model for isotropic turbulence is restated in a form which is independent of the choice of orthogonal basis functions for representing the velocity field. The model is then extended to non-stationary inhomogeneous turbulence with a mean shearing velocity, contained by boundaries of arbitrary shape. A modification of the model is introduced which makes negligible changes in the numerical predictions but which greatly simplifies computations when the covariance matrix and related statistical matrices are non-diagonal. The altered model may be regarded as a kind of generalization of Orszag's eddy-damped Markovian model, with the damping factors determined systematically, in representation-independent form, from dynamical equations. The final equations of the test-field model are presented in a sufficiently explicit form to serve as a starting point for numerical work. To facilitate comparison, the corresponding direct-interaction equations for inhomogeneous turbulence with mean shear are presented also, in a uniform notation. The test-field model is much faster to compute than the direct-interaction approximation because, in the former, only single-time statistical functions need be computed. This advantage is at the cost of a less rich and less faithful representation of the dynamics.

1. Introduction

An approximation for isotropic turbulence, called the test-field model, has been presented in two recent papers (Kraichnan 1971*a*, *b*; Kraichnan 1971*a* will be cited as I). The test-field model was intended to incorporate some characteristic features of the direct-interaction and Lagrangian-history direct-interaction approximations (Kraichnan 1964*a*, 1966), but be simpler to compute. Like the direct-interaction approximation, the test-field model features a generalized Langevin-type amplitude equation, which contains a dynamical damping term and a random driving term. The direct-interaction amplitude equation involves dynamical damping which is non-local in time, and it leads to final statistical equations which are integro-differential in time. The test-field model amplitude equation has a damping term which is local in time and it leads to simultaneous differential equations in time for the energy spectrum function and for certain characteristic memory-time integrals associated with the interaction of wave-number triads. The computational simplicity of the test-field model has its cost in faithfulness of representation, however, The random driving term in this model is a white noise in time and this is basically distasteful, since turbulence generally does not exhibit clearly defined long and short time scales, as do some other problems in statistical dynamics.

The artificiality of the white-noise aspect of the test-field model shows up in a qualitative misrepresentation of two-time correlations and in certain other failings (see I). However, numerical integrations of the equations yield spectral predictions in reasonably good agreement with experiment at both low and high Reynolds numbers (Herring & Kraichnan 1972). The model has also been applied to the growth of uncertainty of knowledge of a turbulent velocity field whose initial state is only partly known (Leith & Kraichnan 1972).

A positive feature of the test-field model is its invariance to random Galilean transformations. In the Lagrangian-history direct-interaction approximation, this invariance is achieved by modifying the direct-interaction approximation so that certain characteristic memory integrals in the latter are taken back along particle trajectories instead of being evaluated at fixed laboratory co-ordinates. This is well-based in the physics but requires an elaborate formalism in which both Eulerian and Lagrangian quantities appear. In the test-field model, a much cruder procedure is used. The characteristic memory times for interaction of wavenumber triads are assumed to be limited by distortion of flow structures, and the distortion is measured by examining the coupling between the transverse and longitudinal components of a pressureless test field which is advected by the turbulence. This procedure, which is motivated and described in detail in I, gives the desired invariance within a purely Eulerian framework and with great simplicity compared with the Lagrangian-history direct-interaction approximation. An advantage of the test-field model is that it does provide an explicit equation of motion for the velocity amplitude, thus ensuring certain realizability properties, while no such model amplitude equation has been found for the Lagrangian-history direct-interaction approximation.

The relative simplicity of the test-field model suggests that an extension to inhomogeneous turbulence and non-zero mean fields may be useful, either as a substitute for the richer direct-interaction approximation, or as a tool for preliminary exploration in numerical computations of the latter. The extension is not immediately obvious. In isotropic turbulence, the covariance matrix of the mode amplitudes is diagonal in the Fourier representation, while in the general case of non-stationary inhomogeneous turbulence there is no time-independent representation in which the covariance matrix is diagonal. This causes no difficulties with the direct-interaction approximation, whose structure is intrinsically representation-independent (Kraichnan 1964*b*). However, the construction of the test-field model in I leaned heavily on the diagonal properties of the Fourier representation. Our procedure now will be first to restate the test-field equations for isotropic turbulence in a form which is independent of representation. Then we shall argue that this same formulation is a proper extension of the model for cases where there is no diagonal representation.

2. Recasting the test-field equations for isotropic turbulence

In order to make the manipulations as transparent as possible, we shall rewrite the basic equations of motion in an abstract notation which displays the dynamical structure in a simple way and also is more compact than the usual wave-vector notation. The incompressible Navier–Stokes equation in a (large) cyclic box has the Fourier representation

$$(\partial/\partial t+\nu k^2)\,u_i(k,t) = -ik_m P_{ij}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} u_j(\mathbf{p},t)\,u_m(\mathbf{q},t), \tag{2.1}$$

where $u_i(\mathbf{k},t)$ is the velocity amplitude and $P_{ij}(\mathbf{k}) = \delta_{ij}-k_i k_j/k^2$. We also need the 'test-field' amplitude equations

$$(\partial/\partial t+\nu k^2)\,v_i^S(\mathbf{k},t) = -ik_m P_{ij}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} v_j^C(\mathbf{p},t)\,\hat{u}_m(\mathbf{q},t), \tag{2.2}$$

$$(\partial/\partial t+\nu k^2)\,v_i^C(\mathbf{k},t) = -ik_m \Pi_{ij}(\mathbf{k}) \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} v_j^S(\mathbf{p},t)\,\hat{u}_m(\mathbf{q},t), \tag{2.3}$$

where $\mathbf{v}^S$ and $\mathbf{v}^C$ are, respectively, the solenoidal and longitudinal parts of a test field advected by the solenoidal field $\hat{\mathbf{u}}$, with $\Pi_{ij}(\mathbf{k}) = k_i k_j/k^2$. $P_{ij}(\mathbf{k})$ and $\Pi_{ij}(\mathbf{k})$ are transverse and longitudinal projection operators, respectively. In (2.2) and (2.3), only that part of the advection coupling which couples $\mathbf{v}^S$ and $\mathbf{v}^C$ to each other is retained. The field $\hat{\mathbf{u}}$ is a random solenoidal field whose single-time covariance equals that of the turbulent velocity field $\mathbf{u}$.

For each allowed $\mathbf{k}$, let us introduce three mutually orthogonal unit vectors $\mathbf{n}^i(\mathbf{k})$ such that $\mathbf{n}^1(\mathbf{k})$ and $\mathbf{n}^2(\mathbf{k})$ lie in the plane normal to $\mathbf{k}$, while $\mathbf{n}^3(\mathbf{k})$ is parallel to $\mathbf{k}$. We require $\mathbf{n}^i(-\mathbf{k}) = \mathbf{n}^i(\mathbf{k})$. The choice of the unit vectors is otherwise arbitrary. The field $\mathbf{u}(\mathbf{k},t)$ can now be described by components along these vectors. Let all these components be arranged in a single linear sequence and denoted by $u_\alpha(t)$, where α takes all integer values. The correspondence $\alpha\to(i,\mathbf{k})$, where i is the unit-vector label, is arbitrary except for the restriction that if $\alpha\to(i,\mathbf{k})$ then $-\alpha\to(i,-\mathbf{k})$. Since $\mathbf{u}$ is solenoidal, every third component (those for $i=3$) vanishes. We make the same decomposition for the remaining vector fields, thereby obtaining the variables $v_\alpha^S(t)$, $v_\alpha^C(t)$ and $\hat{u}_\alpha(t)$. All components $v_\alpha^S(t)$ and $\hat{u}_\alpha(t)$ corresponding to $i=3$ vanish, as do all components $v_\alpha^C(t)$ corresponding to $i=1,2$.

Equations (2.1)–(2.3) readily yield equations of motion for the new variables:

$$(d/dt+\nu_\alpha)\,u_\alpha = \Sigma_{\beta\gamma} A_{-\alpha\beta\gamma} u_\beta u_\gamma, \tag{2.4}$$

$$(d/dt+\nu_\alpha)\,v_\alpha^S = -\Sigma_{\beta\gamma} B_{\beta,-\alpha\gamma} v_\beta^C \hat{u}_\gamma, \tag{2.5}$$

$$(d/dt+\nu_\alpha)\,v_\alpha^C = \Sigma_{\beta\gamma} B_{-\alpha\beta\gamma} v_\beta^S \hat{u}_\gamma. \tag{2.6}$$

Here the ν_α are the appropriate values of νk^2. If

$$\alpha\to(i,\mathbf{k}),\quad \beta\to(j,\mathbf{p}),\quad \gamma\to(m,\mathbf{q}),$$

the coefficients A and B are easily found to be

$$A_{\alpha\beta\gamma} = i[\mathbf{n}^i(\mathbf{k}).\mathbf{n}^j(\mathbf{p})]\,[\mathbf{k}.\mathbf{n}^m(\mathbf{q})]\,\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})\quad (i,j,m=1,2), \tag{2.7}$$

$$B_{\alpha\beta\gamma} = i[\mathbf{n}^i(\mathbf{k}).\mathbf{n}^j(\mathbf{p})]\,[\mathbf{k}.\mathbf{n}^m(\mathbf{q})]\,\delta(\mathbf{k}+\mathbf{p}+\mathbf{q})\quad (i=3;j,m=1,2), \tag{2.8}$$

where $\delta(\mathbf{k}) = 1$ for $\mathbf{k} = 0$ and 0 for $\mathbf{k} \neq 0$. The coefficients vanish for other combinations of i, j, m values.

These coefficients satisfy

$$A_{-\alpha,-\beta,-\gamma} = A^*_{\alpha\beta\gamma}, \quad B_{-\alpha,-\beta,-\gamma} = B^*_{\alpha\beta\gamma}, \tag{2.9}$$

so that (2.4)–(2.6) preserve the reality condition $u_{-\alpha}(t) = u^*_\alpha(t)$, etc. Noting that $\mathbf{q}.\mathbf{n}^m(\mathbf{q}) = 0$ $(m = 1, 2)$ [a fact already used in writing (2.5)], we find that

$$A_{\alpha\beta\gamma} + A_{\beta\alpha\gamma} = 0, \tag{2.10}$$

whence (2.4) gives conservation of $\Sigma_\alpha |u_\alpha(t)|^2$ (twice the kinetic energy per unit mass) by the nonlinear terms. Similarly, the nonlinear terms in (2.5) and (2.6) conserve $\Sigma_\alpha(|v^S_\alpha(t)|^2 + |v^C_\alpha(t)|^2)$. If we write

$$\bar{A}_{\alpha\beta\gamma} = A_{\alpha\beta\gamma} + A_{\alpha\gamma\beta},$$

then (2.10) gives

$$\bar{A}_{\alpha\beta\gamma} + \bar{A}_{\beta\gamma\alpha} + \bar{A}_{\gamma\alpha\beta} = 0, \tag{2.11}$$

an alternative form of the conservation identity. Note that only the symmetrized coefficient $\bar{A}$ can actually contribute to the right-hand side of (2.4).

The covariance and response matrices are defined by

$$U_{\alpha\beta}(t,t') = \langle u_\alpha(t)\, u^*_\beta(t')\rangle = \langle u_\alpha(t)\, u_{-\beta}(t')\rangle,$$
$$G_{\alpha\beta}(t,t') = \langle \delta u_\alpha(t)/\delta f_\beta(t')\rangle \quad (t \geqslant t'),$$

where $\delta f_\alpha(t)$ is an infinitesimal random driving term added to the right-hand side of (2.4). When the turbulence is statistically homogeneous and isotropic we have

$$U_{\alpha\beta}(t,t') = \delta_{\alpha\beta} U_{\alpha\alpha}(t,t'), \quad G_{\alpha\beta}(t,t') = \delta_{\alpha\beta} G_{\alpha\alpha}(t,t'). \tag{2.12}$$

The test-field model as presented in I applies to this diagonal situation. The basic equations of the model in the present notation are as follows. The Langevin-type amplitude equations which model (2.4)–(2.6) are

$$[d/dt + \eta_{\alpha\alpha}(t)]\, u_\alpha(t) = w(t)\, \Sigma_{\beta\gamma} A_{-\alpha\beta\gamma} [\theta_{\alpha\beta\gamma}(t)]^{\frac{1}{2}}\, \xi_\beta(t)\, \xi_\gamma(t), \tag{2.13}$$
$$[d/dt + \eta^S_{\alpha\alpha}(t)]\, v^S_\alpha(t) = -w(t)\, \Sigma_{\beta\gamma} B_{\beta,-\alpha\gamma} [\theta^G_{\beta\alpha\gamma}(t)]^{\frac{1}{2}}\, \xi^C_\beta(t)\, \hat{\xi}_\gamma(t), \tag{2.14}$$
$$[d/dt + \eta^C_{\alpha\alpha}(t)]\, v^C_\alpha(t) = w(t)\, \Sigma_{\beta\gamma} B_{-\alpha\beta\gamma} [\theta^G_{\alpha\beta\gamma}(t)]^{\frac{1}{2}}\, \xi^S_\beta(t)\, \hat{\xi}_\gamma(t), \tag{2.15}$$

with the total (viscous + dynamical) damping factors given by

$$\eta_{\alpha\alpha}(t) = \nu_\alpha - \Sigma_{\beta\gamma} \bar{A}_{-\alpha\beta\gamma} \bar{A}_{-\beta\alpha,-\gamma}\, \theta_{\alpha\beta\gamma}(t)\, U_{\gamma\gamma}(t), \tag{2.16}$$
$$\eta^S_{\alpha\alpha}(t) = \nu_\alpha + \Sigma_{\beta\gamma} |B_{\beta,-\alpha\gamma}|^2\, \theta^G_{\beta\alpha\gamma}(t)\, U_{\gamma\gamma}(t), \tag{2.17}$$
$$\eta^C_{\alpha\alpha}(t) = \nu_\alpha + \Sigma_{\beta\gamma} |B_{-\alpha\beta\gamma}|^2\, \theta^G_{\alpha\beta\gamma}(t)\, U_{\gamma\gamma}(t), \tag{2.18}$$

where $U_{\gamma\gamma}(t) = U_{\gamma\gamma}(t,t)$. Here $w(t)$ is a white-noise process which satisfies

$$\langle w(t)\, w(t')\rangle = 2\delta(t-t'),$$

while the ξ's are Gaussian random variables which satisfy

$$\langle \xi_\alpha(t)\, \xi^*_\beta(t')\rangle = \langle \hat{\xi}_\alpha(t)\, \hat{\xi}^*_\beta(t')\rangle = U_{\alpha\beta}(t,t'),$$
$$\langle \xi^S_\alpha(t)\, \xi^S_\beta(t')\rangle = \langle v^S_\alpha(t)\, v^S_\beta(t')\rangle, \; \langle \xi^C_\alpha(t)\, \xi^C_\beta(t')\rangle = \langle v^C_\alpha(t)\, v^C_\beta(t')\rangle, \tag{2.19}$$

but are otherwise uncorrelated. The θ's are characteristic memory times for three-mode interactions and are given by

$$\theta_{\alpha\beta\gamma}(t) = \int_0^t G^S_{\alpha\alpha}(t,s)\, G^S_{\beta\beta}(t,s)\, G^S_{\gamma\gamma}(t,s)\, ds,$$

$$\theta^G_{\alpha\beta\gamma}(t) = \int_0^t G^C_{\alpha\alpha}(t,s)\, G^S_{\beta\beta}(t,s)\, G^S_{\gamma\gamma}(t,s)\, ds, \tag{2.20}$$

where $G^S_{\alpha\beta}(t,s)$ and $G^C_{\alpha\beta}(t,s)$ are the response matrices associated with (2.14) and (2.15). In the present case these matrices are diagonal and satisfy ($t \geqslant t'$)

$$[d/dt + \eta^S_{\alpha\alpha}(t)]\, G^S_{\alpha\alpha}(t,t') = 0, \quad G^S_{\alpha\alpha}(t',t') = 1, \tag{2.21}$$

$$[d/dt + \eta^C_{\alpha\alpha}(t)]\, G^C_{\alpha\alpha}(t,t') = 0, \quad G^C_{\alpha\alpha}(t',t') = 1. \tag{2.22}$$

Equations (2.13)–(2.19) imply that

$$[d/dt + 2\eta_{\alpha\alpha}(t)]\, U_{\alpha\alpha}(t) = \Sigma_{\beta\gamma} |\bar{A}_{-\alpha\beta\gamma}|^2\, \theta_{\alpha\beta\gamma}(t)\, U_{\beta\beta}(t)\, U_{\gamma\gamma}(t), \tag{2.23}$$

which gives the evolution of the modal intensities. Differentiation of (2.20), and use of (2.21) and (2.22), gives evolution equations for the memory times:

$$d\theta_{\alpha\beta\gamma}(t)/dt = 1 - [\eta^S_{\alpha\alpha}(t) + \eta^S_{\beta\beta}(t) + \eta^S_{\gamma\gamma}(t)]\, \theta_{\alpha\beta\gamma}(t),$$

$$d\theta^G_{\alpha\beta\gamma}(t)/dt = 1 - [\eta^C_{\alpha\alpha}(t) + \eta^S_{\beta\beta}(t) + \eta^S_{\gamma\gamma}(t)]\, \theta^G_{\alpha\beta\gamma}(t). \tag{2.24}$$

Equations (2.16)–(2.18), (2.23) and (2.24) are a complete set for the determination of $U_{\alpha\alpha}(t)$.

The derivation of (2.13)–(2.24) follows straightforwardly from that given in I, using conventional wave-vector notation, and we shall save space by not repeating the arguments here.† The following properties of the final equations should be noted. By its definition, $U_{\alpha\alpha}(t)$ is real and positive, and $U_{\alpha\alpha}(t) = U_{-\alpha,-\alpha}(t)$. It then follows from (2.17), (2.18) and (2.20)–(2.22), together with (2.9), that $\eta^S_{\alpha\alpha}(t), \eta^C_{\alpha\alpha}(t), G^S_{\alpha\alpha}(t,t')$ and $G^C_{\alpha\alpha}(t,t')$ also are real, positive, and unchanged for $\alpha \to -\alpha$. Moreover, $G^S_{\alpha\alpha}(t,t')$ and $G^C_{\alpha\alpha}(t,t')$ are monotonically decreasing functions of $t-t'$. Equation (2.23) is identically conservative, apart from viscous dissipation, in consequence of (2.11). The right-hand side of (2.23) contains only positive contributions. It is clear from (2.11) that the dynamic contribution to (2.16) from any given triad interaction is typically positive, although every contribution need not always be so. It is easy to see from the symmetries of the A's that $\eta_{-\alpha,-\alpha}(t) = \eta^*_{\alpha\alpha}(t)$, while the reality of $\eta_{\alpha\alpha}(t)$ follows immediately from the fact that the A's are pure imaginary while all other factors in (2.16) are real.

3. Non-diagonal generalization

The extension of the test-field model to non-diagonal cases consists of two stages. First, we shall rewrite the diagonal equations in a form which is invariant to a complex rotation in the space of the u_α. This assures that results obtained

† Equation (3.1) of I, which corresponds to (2.13), involves two sets of variables ξ and ξ'. This difference does not affect the final statistical equations. A further difference is that external forcing is omitted in the present paper. [Note that a factor $2^{-\frac{1}{2}}$ is missing from the right-hand sides of equations (2.4) and (3.3) of I.]

from the model are independent of which representation the equations are solved in. The second stage, which will require discussion, is to use the representation-independent equations even when there is no representation that fully diagonalizes the matrices.

If we set $\nu_{\alpha\beta} = \delta_{\alpha\beta}\nu_\alpha$, (2.4)–(2.6) can be rewritten as

$$du_\alpha/dt + \nu_{\alpha\beta}u_\beta = A_{-\alpha\beta\gamma}u_\beta u_\gamma, \tag{3.1}$$

$$dv_\alpha^S/dt + \nu_{\alpha\beta}v_\beta^S = -B_{\beta,-\alpha\gamma}v_\beta^C \hat{u}_\gamma, \tag{3.2}$$

$$dv_\alpha^C/dt + \nu_{\alpha\beta}v_\beta^C = B_{-\alpha\beta\gamma}v_\beta^S \hat{u}_\gamma. \tag{3.3}$$

In (3.1)–(3.3), *and hereafter unless stated otherwise, the summation convention is used on repeated Greek indices.*

Now consider the unitary transformation

$$u_\alpha \to O_{\alpha\beta}u_\beta, \quad v_\alpha^S \to O_{\alpha\beta}v_\beta^S, \quad v_\alpha^C \to O_{\alpha\beta}v_\beta^C, \quad \hat{u}_\alpha \to O_{\alpha\beta}\hat{u}_\beta, \tag{3.4}$$

where O satisfies

$$O_{\alpha\beta}^{-1} = O_{\beta\alpha}^*, \quad O_{\alpha\gamma}O_{\beta\gamma}^* = O_{\gamma\alpha}O_{\gamma\beta}^* = \delta_{\alpha\beta}, \quad O_{-\alpha,-\beta} = O_{\alpha\beta}^*. \tag{3.5}$$

This transformation leaves $u_\alpha u_\alpha^*$, $v_\alpha^S v_\alpha^{S*} + v_\alpha^C v_\alpha^{C*}$ and $\hat{u}_\alpha \hat{u}_\alpha^*$ invariant and preserves the reality property $u_{-\alpha} = u_\alpha^*$, etc. Moreover, (3.1)–(3.3) remain invariant in form, the coefficients transforming according to

$$\nu_{\alpha\beta} \to O_{\alpha\mu}\nu_{\mu\epsilon}O_{\epsilon\beta}^{-1} = O_{\alpha\mu}O_{\beta\epsilon}^*\nu_{\mu\epsilon}, \quad A_{\alpha\beta\gamma} \to O_{\alpha\mu}^*O_{\beta\epsilon}^*O_{\gamma\lambda}^*A_{\mu\epsilon\lambda},$$
$$B_{\alpha\beta\gamma} \to O_{\alpha\mu}^*O_{\beta\epsilon}^*O_{\mu\lambda}^*B_{\mu\epsilon\lambda},$$

where we use all the properties (3.5). Note that this transformation preserves (2.9)–(2.11). The transformations of covariance and response matrices are (we specialize now to transformations O which do not mix transverse and longitudinal components)

$$U_{\alpha\beta}(t,t') \to O_{\alpha\mu}O_{\beta\epsilon}^*U_{\mu\epsilon}(t,t'),$$
$$G_{\alpha\beta}^S(t,t') \to O_{\alpha\mu}G_{\mu\epsilon}^S(t,t')O_{\epsilon\beta}^{-1} = O_{\alpha\mu}O_{\beta\epsilon}^*G_{\mu\epsilon}^S(t,t'),$$

with a corresponding rule for G^C. $U_{\alpha\beta}(t,t')$ is Hermitian by definition and this property is preserved under the transformation. Clearly, if G^S and G^C are Hermitian (to be discussed later), this also is preserved.

The principal task in writing (2.13)–(2.15) in representation-independent form comes from the $\theta^{\frac{1}{2}}$ factors, which must be replaced in a consistent way by matrix square roots. Accordingly, we define the matrices

$$\theta_{\alpha\beta\gamma\mu\epsilon\lambda}(t) = \int_0^t G_{\alpha\mu}^S(t,s)\,G_{\beta\epsilon}^S(t,s)\,G_{\gamma\lambda}^S(t,s)\,ds,$$

$$\theta_{\alpha\beta\gamma\mu\epsilon\lambda}^G(t) = \int_0^t G_{\alpha\mu}^C(t,s)\,G_{\beta\epsilon}^S(t,s)\,G_{\gamma\lambda}^S(t,s)\,ds, \tag{3.6}$$

and define square-root matrices by

$$h_{\alpha\beta\gamma\mu\epsilon\lambda}(t)\,h_{\mu\epsilon\lambda\tau\nu\pi}(t) = \theta_{\alpha\beta\gamma\tau\nu\pi}(t),$$
$$h_{\alpha\beta\gamma\mu\epsilon\lambda}^G(t)\,h_{\mu\epsilon\lambda\tau\nu\pi}^G(t) = \theta_{\alpha\beta\gamma\tau\nu\pi}^G(t). \tag{3.7}$$

The laws

$$\theta_{\alpha\beta\gamma\mu\epsilon\lambda}(t) \to O_{\alpha\tau} O_{\beta\nu} O_{\gamma\pi} O^*_{\mu\sigma} O^*_{\epsilon\delta} O^*_{\lambda\omega} \theta_{\tau\nu\pi\sigma\delta\omega}(t),$$
$$h_{\alpha\beta\gamma\mu\epsilon\lambda}(t) \to O_{\alpha\tau} O_{\beta\nu} O_{\gamma\pi} O^*_{\mu\sigma} O^*_{\epsilon\delta} O^*_{\lambda\omega} h_{\tau\nu\pi\sigma\delta\omega}(t),$$

and corresponding rules for θ^G and h^G, follow from (3.6), (3.7) and the transformation rule for G^S and G^C. In a representation where G^S and G^C are diagonal,

$$h_{\alpha\beta\gamma\mu\epsilon\lambda}(t) = \delta_{\alpha\mu}\delta_{\beta\epsilon}\delta_{\gamma\lambda}[\theta_{\alpha\beta\gamma}(t)]^{\frac{1}{2}},$$
$$h^G_{\alpha\beta\gamma\mu\epsilon\lambda}(t) = \delta_{\alpha\mu}\delta_{\beta\epsilon}\delta_{\gamma\lambda}[\theta^G_{\alpha\beta\gamma}(t)]^{\frac{1}{2}} \quad \text{(no summations)}. \tag{3.8}$$

We shall save for later the question of existence and uniqueness of h and h^G in the general case where there may be no time-independent diagonal representation.

Let us define the effective interaction coefficients

$$C_{\alpha\beta\gamma}(t) = A_{\mu\epsilon\lambda} h_{\mu\epsilon\lambda\alpha\beta\gamma}(t), \quad D_{\alpha\beta\gamma}(t) = B_{\mu\epsilon\lambda} h^G_{\mu\epsilon\lambda\alpha\beta\gamma}(t). \tag{3.9}$$

$C_{\alpha\beta\gamma}(t)$ and $D_{\alpha\beta\gamma}(t)$ obey the same symmetry and conservation identities as do $A_{\alpha\beta\gamma}$ and $B_{\alpha\beta\gamma}$ [equations (2.9)–(2.11)]. We can now replace the test-field model equations of § 2 by the following set of matrix equations:

$$du_\alpha(t) + \eta_{\alpha\beta}(t)\, u_\beta(t) = w(t)\, C_{-\alpha\beta\gamma}(t)\, \xi_\beta(t)\, \xi_\gamma(t), \tag{3.10}$$
$$dv^S_\alpha(t) + \eta^S_{\alpha\beta}(t)\, v^S_\beta(t) = -w(t)\, D_{\beta,-\alpha\gamma}(t)\, \xi^C_\beta(t)\, \hat{\xi}_\gamma(t), \tag{3.11}$$
$$dv^C_\alpha(t) + \eta^C_{\alpha\beta}(t)\, v^C_\beta(t) = w(t)\, D_{-\alpha\beta\gamma}(t)\, \xi^S_\beta(t)\, \hat{\xi}_\gamma(t). \tag{3.12}$$

Here the ξ variables are defined as before and the damping matrices are given by

$$\eta_{\alpha\beta}(t) = \nu_{\alpha\beta} - \bar{C}_{-\alpha\mu\gamma}(t)\, \bar{C}^*_{\mu,-\beta\epsilon}(t)\, U_{\gamma\epsilon}(t), \tag{3.13}$$
$$\eta^S_{\alpha\beta}(t) = \nu^S_{\alpha\beta} + D_{\mu,-\alpha\gamma}(t)\, D^*_{\mu,-\beta\epsilon}(t)\, U_{\gamma\epsilon}(t), \tag{3.14}$$
$$\eta^C_{\alpha\beta}(t) = \nu^C_{\alpha\beta} + D_{-\alpha\mu\gamma}(t)\, D^*_{-\beta\mu\epsilon}(t)\, U_{\gamma\epsilon}(t), \tag{3.15}$$

where
$$\bar{C}_{\alpha\beta\gamma}(t) = C_{\alpha\beta\gamma}(t) + C_{\alpha\gamma\beta}(t), \quad U_{\gamma\epsilon}(t) = U_{\gamma\epsilon}(t,t),$$

and property (2.9) is used. Here $\nu_{\alpha\beta} = \nu^S_{\alpha\beta} + \nu^C_{\alpha\beta}$, where $\nu^S_{\alpha\beta}(\nu^C_{\alpha\beta})$ is pure solenoidal (longitudinal).

The matrix equations of motion for U, G^S, G^C, θ and θ^G are

$$dU_{\alpha\beta}/dt + \eta_{\alpha\gamma} U_{\gamma\beta} + \eta^*_{\beta\gamma} U_{\alpha\gamma} = \bar{C}_{-\alpha\mu\gamma} \bar{C}^*_{-\beta\epsilon\lambda} U_{\mu\epsilon} U_{\gamma\lambda}, \tag{3.16}$$
$$dG^S_{\alpha\beta}(t,t') + \tfrac{1}{2}[\eta^S_{\alpha\gamma}(t)\, G^S_{\gamma\beta}(t,t') + G^S_{\alpha\gamma}(t,t')\, \eta^S_{\gamma\beta}(t)] = 0 \quad (t \geqslant t'). \tag{3.17}$$
$$dG^C_{\alpha\beta}(t,t') + \tfrac{1}{2}[\eta^C_{\alpha\gamma}(t)\, G^C_{\gamma\beta}(t,t') + G^C_{\alpha\gamma}(t,t')\, \eta^C_{\gamma\beta}(t)] = 0 \quad (t \geqslant t'), \tag{3.18}$$

$$\begin{aligned} d\theta_{\alpha\beta\gamma\mu\epsilon\lambda}/dt + \tfrac{1}{2}(\eta^S_{\alpha\tau}\theta_{\tau\beta\gamma\mu\epsilon\lambda} + \eta^S_{\beta\tau}\theta_{\alpha\tau\gamma\mu\epsilon\lambda} + \eta^S_{\gamma\tau}\theta_{\alpha\beta\tau\mu\epsilon\lambda} \\ + \theta_{\alpha\beta\gamma\tau\epsilon\lambda}\eta^S_{\tau\mu} + \theta_{\alpha\beta\gamma\mu\tau\lambda}\eta^S_{\tau\epsilon} + \theta_{\alpha\beta\gamma\mu\epsilon\tau}\eta^S_{\tau\lambda}) = \delta_{\alpha\mu}\delta_{\beta\epsilon}\delta_{\gamma\lambda}, \end{aligned} \tag{3.19}$$

$$\begin{aligned} d\theta^G_{\alpha\beta\gamma\mu\epsilon\lambda}/dt + \tfrac{1}{2}(\eta^C_{\alpha\tau}\theta^G_{\tau\beta\gamma\mu\epsilon\lambda} + \eta^S_{\beta\tau}\theta^G_{\alpha\tau\gamma\mu\epsilon\lambda} + \eta^S_{\gamma\tau}\theta^G_{\alpha\beta\tau\mu\epsilon\lambda} \\ + \theta^G_{\alpha\beta\gamma\tau\epsilon\lambda}\eta^C_{\tau\mu} + \theta^G_{\alpha\beta\gamma\mu\tau\lambda}\eta^S_{\tau\epsilon} + \theta^G_{\alpha\beta\gamma\mu\epsilon\tau}\eta^S_{\tau\lambda}) = \delta_{\alpha\mu}\delta_{\beta\epsilon}\delta_{\gamma\lambda}. \end{aligned} \tag{3.20}$$

All quantities in (3.16), (3.19) and (3.20) have argument t.

Equations (3.7), (3.9), (3.13)–(3.16), (3.19) and (3.20) are a complete set for determination of $U_{\alpha\beta}(t)$ from given initial values $U_{\alpha\beta}(0)$. The initial conditions on the memory-time matrices are $\theta_{\alpha\beta\gamma\mu\epsilon\lambda}(0) = 0$ and $\theta^G_{\alpha\beta\gamma\mu\epsilon\lambda}(0) = 0$. For isotropic turbulence, the matrices U, G^S, G^C, η, η^S and η^C are all diagonal in the representation we have adopted, and these equations reduce to the diagonal test-field equations of §2. Also, it is clear that the present set preserves the diagonal property. This is obvious if the damping matrices stay diagonal. For the latter, diagonality in wavenumber follows from the δ factors in (2.7) and (2.8), while diagonality in the unit-vector indices is a consequence of isotropy, which is not violated by any of our manipulations.

The matrix test-model equations just given are invariant to all transformations (3.4), (3.5); that is, to transformations which leave $u_\alpha u_\alpha^*$ invariant. This means, that no matter what representation the equations are solved in, the results, transformed back to the diagonal representation, are the same as if the diagonal test-field equations had been solved. What, now, of cases where there is *no* representation that diagonalizes all the matrices for all t? First, it is easy to verify, using the representation-invariant properties (2.9) that hermiticity of U, G^S and G^C is preserved under the equations of motion and that η^S and η^C are Hermitian. The symmetrized form adopted for (3.17) and (3.18) is taken expressly to assure that G^S and G^C are Hermitian. When there is a diagonal representation, G^S and G^C are the response matrices of (3.11) and (3.12), in any representation. When no diagonal representation valid for all t exists, we regard (3.17) and (3.18) as defining G^S and G^C, a procedure which does not substantially add to the considerable arbitrariness of the test-field model. The hermiticity of G^S and G^C implies that the memory-time matrices are Hermitian:

$$\theta_{\mu\epsilon\lambda\alpha\beta\gamma}(t) = \theta^*_{\alpha\beta\gamma\mu\epsilon\lambda}(t), \quad \theta^G_{\mu\epsilon\lambda\alpha\beta\gamma}(t) = \theta^{G*}_{\alpha\beta\ \gamma\mu\epsilon\lambda}(t). \tag{3.21}$$

The crucial point in establishing self-consistency of the matrix equations is demonstrating that the eigenvalues of θ and θ^G are all non-negative, so that the square-root matrices h and h^G can be uniquely defined and have non-negative eigenvalues. This is easily shown by first showing that the eigenvalues of G^S and G^C are all non-negative, a fact which can be demonstrated by induction. Suppose that $G^S_{\alpha\beta}(t,t')$ has non-negative eigenvalues, and transform to a representation where it is diagonal (always possible for any given t and t'). Then, to order Δt,

$$\begin{aligned} G^S_{\alpha\alpha}(t+\Delta t, t') &= [1-\Delta t\eta^S_{\alpha\alpha}(t)]\,G^S_{\alpha\alpha}(t,t'),\\ G^S_{\alpha\beta}(t+\Delta t, t') &= -\tfrac{1}{2}\Delta t(\eta^S_{\alpha\beta}(t)\,G^S_{\beta\beta}(t,t') + G^S_{\alpha\alpha}(t,t')\,\eta^S_{\alpha\beta}(t)] \quad (\alpha \neq \beta), \end{aligned} \tag{3.22}$$

where the summation convention is suspended. Since the off-diagonal elements of $G^S(t+\Delta t, t')$ are of $O(\Delta t)$, the unitary transformation which re-diagonalizes G^S at $t+\Delta t$ differs from the unit matrix by $O(\Delta t)$. This implies that the eigenvalues of $G^S(t+\Delta t, t')$ differ from the diagonal elements given in (3.22) only by $O(\Delta t^2)$. Taking the limit $\Delta t \to 0$, we see that the eigenvalues of $G^S(t,t')$ remain real and can never change sign as a function of t; they vary according to

$$\lambda_\alpha(t,t') = \exp[\mu_\alpha(t,t')],$$

where λ_α is an eigenvalue and μ_α is some real function. $G^S(t,'t')$ is the unit matrix, so that $\mu_\alpha(t',t') = 0$. Next we note that the diagonal elements of $\eta^S(t)$ are non-negative, which follows from the form of (3.14), the fact that, by definition, ν and U have non-negative eigenvalues and the lemma that the diagonal elements of a Hermitian matrix with non-negative eigenvalues are non-negative in any representation.† This shows that $\mu_\alpha(t,t')$ and $\lambda_\alpha(t,t')$ are monotonically decreasing functions of t. The same analysis holds for G^C.

A consistency point here is that (3.16) preserves positivity of the eigenvalues of $U(t)$. This follows from the fact that (3.16) is an exact consequence of the amplitude equation (3.10) [cf. I].

Finally, now, we have that the integrands of (3.6) have all positive eigenvalues so that, by the preceding lemma, θ and θ^G have all positive eigenvalues. We may therefore uniquely specify the matrices h and h^G, obeying (3.7), as the matrices whose eigenvalues are the positive square roots of the eigenvalues of the matrices θ and θ^G respectively. Thus the matrices U, η^S, η^C, G^S, G^C, θ, θ^G, h and h^G all are Hermitian with non-negative eigenvalues. On the other hand, η need not be Hermitian, in general.

As we have stated before, the matrix test-field equations are invariant to transformations which leave $u_\alpha u_\alpha^*$ invariant, the only change being the introduction of new values of the A and B coefficients according to the rules stated after (3.5). Particular such transformations are from Fourier space back to x space and from Fourier or x space to, say, Legendre-polynomial decomposition of the fields (Orszag 1971). Within the Fourier representation, we can transform from our present decomposition, with coefficients (2.7) and (2.8), to one into components of positive and negative helicity, or, most simply, we can transform back to ordinary vector components in fixed Cartesian co-ordinates. In the latter case, we again make the identification

$$\alpha \to (i,\mathbf{k}), \quad \beta \to (j,\mathbf{p}), \quad \gamma \to (m,\mathbf{q}),$$

where now i, j and m are ordinary tensor indices instead of unit-vector labels. The new coefficients are

$$\begin{aligned} A_{\alpha\beta\gamma} &= ik_s P_{ir}(\mathbf{k})\, P_{rj}(\mathbf{p})\, P_{sm}(\mathbf{q})\, \delta(\mathbf{k}+\mathbf{p}+\mathbf{q}), \\ B_{\alpha\beta\gamma} &= ik_s \Pi_{ir}(\mathbf{k})\, P_{rj}(\mathbf{p})\, P_{sm}(\mathbf{q})\, \delta(\mathbf{k}+\mathbf{p}+\mathbf{q}). \end{aligned} \tag{3.23}$$

The extra P factors in (3.23), compared with (2.1)–(2.3), express the fact that coefficients corresponding to longitudinal components of u, $\hat{u}$ and v^S vanish. The construction of h and h^G in this representation is facilitated by the fact that $P_{ij}(\mathbf{k})$ and $\Pi_{ij}(\mathbf{k})$ are their own matrix square roots.

4. Comparison with the direct-interaction equations

Since the test-field model for isotropic turbulence was developed from the direct-interaction approximation, it is of interest to compare the present matrix form of the test-field model with the direct-interaction equations, expressed in the same notation. The direct-interaction approximation for inhomogeneous turbulence has been treated in some previous papers (Kraichnan 1964*b*,*c*; Herring 1969). In our present notation, the direct-interaction model amplitude equation is

$$du_\alpha(t)/dt + \int_0^t \eta_{\alpha\beta}(t,s)\, u_\beta(s)\, ds = A_{-\alpha\beta\gamma}\, \xi_\beta(t)\, \xi_\gamma(t), \tag{4.1}$$

† If the eigenvalues λ_α of a matrix M are all real and positive, the transformed matrix $O_{\alpha\gamma}\lambda_\gamma O^{-1}_{\gamma\beta} = O_{\alpha\gamma}O^*_{\beta\gamma}\lambda_\gamma$ obviously has only non-negative diagonal elements.

where ξ is defined as before and

$$\eta_{\alpha\beta}(t,s) = \nu_{\alpha\beta} - \bar{A}_{-\alpha\mu\gamma}\bar{A}^*_{\lambda,-\beta\epsilon}G_{\mu\lambda}(t,s)\,U_{\gamma\epsilon}(t,s). \tag{4.2}$$

Note that the dynamical damping here is one with memory and that there is no white-noise process in (4.1). Equation (4.1) yields the following equations for the evolution of the single-time covariance, two-time covariance and response matrix (here the response matrix of (4.1) itself):

$$dU_{\alpha\beta}(t)/dt + \int_0^t [\eta_{\alpha\gamma}(t,s)\,U_{\gamma\beta}(s,t) + \eta^*_{\beta\gamma}(t,s)\,U_{\alpha\gamma}(t,s)]\,ds$$
$$= \int_0^t [A_{-\alpha\mu\gamma}\bar{A}^*_{-\tau\epsilon\lambda}G^*_{\beta\tau}(t,s)\,U_{\mu\epsilon}(t,s)\,U_{\gamma\lambda}(t,s)$$
$$+ A^*_{-\beta\epsilon\lambda}\bar{A}_{-\tau\mu\gamma}G_{\alpha\tau}(t,s)\,U_{\mu\epsilon}(s,t)\,U_{\gamma\lambda}(s,t)]\,ds, \tag{4.3}$$

$$\partial U_{\alpha\beta}(t,t')/\partial t + \int_0^t \eta_{\alpha\gamma}(t,s)\,U_{\gamma\beta}(s,t')\,ds = \int_0^{t'} A_{-\alpha\mu\gamma}\bar{A}^*_{-\tau\epsilon\lambda}G^*_{\beta\tau}(t',s)\,U_{\mu\epsilon}(t,s)\,U_{\gamma\lambda}(t,s)\,ds, \tag{4.4}$$

$$\partial G_{\alpha\beta}(t,t')/\partial t + \int_{t'}^t \eta_{\alpha\gamma}(t,s)\,G_{\gamma\beta}(s,t')\,ds = 0 \quad (t \geqslant t'). \tag{4.5}$$

The two approximations have several features in common. Since both direct-interaction and test-field approximations are based on a model amplitude equation, they both assure realizable $U_{\alpha\beta}(t,t')$. In particular, $U_{\alpha\beta}(t)$ has non-negative eigenvalues (positivity of energy spectrum). The integral terms in (4.3) represent the direct-interaction approximation for $A_{-\alpha\mu\gamma}\langle u_\mu u_\gamma u^*_\beta\rangle$ + complex conjugate. Notice that the direct-interaction expression for $\langle u_\mu u_\gamma u^*_\beta\rangle$ then evidently involves couplings with mode triads $(\tau\epsilon\lambda)$ which, in general, differ from $(\mu\gamma\beta)$. This is in contrast to the diagonal (isotropic) case, where the direct-interaction triple-moment results involve explicitly only the coupling coefficients of the three modes immediately concerned. In the test-field model, the same qualitative kind of mixing of interaction coefficients is effected by (3.9).

As in the isotropic case, the matrix test-field equations yield time-displaced covariances $U_{\alpha\beta}(t,t')$ which have qualitatively incorrect features, notably a cusp-like behaviour at $t = t'$. This is in contrast to the direct-interaction results, which are qualitatively correct in their $t-t'$ dependence. The direct-interaction set (4.2)–(4.5) must all be solved as a coupled system of integro-differential equations in time. In contrast, the complete test-field set of § 3 involved only single-time quantities. The test-field equation for $U_{\alpha\beta}(t,t')$ is

$$\partial U_{\alpha\beta}(t,t')/\partial t + \eta_{\alpha\gamma}(t)\,U_{\gamma\beta}(t,t') = 0 \quad (t \geqslant t'), \tag{4.6}$$

with $U_{\alpha\beta}(t',t) = U^*_{\beta\alpha}(t,t')$. This follows directly from (3.10) (cf. I). Equation (4.6) can be solved after $U_{\alpha\beta}(t)$ and $\eta_{\alpha\beta}(t)$ have been determined.

We shall discuss the relative computational efforts for direct-interaction and test-field approximations, and the question of random Galilean invariance, in the sections which follow.

5. Non-zero mean velocity and generalized boundary conditions

The analysis so far assumes implicitly that there is no mean velocity field. In the case of isotropic or reflexion-invariant homogeneous turbulence, preservation of this condition under the equations of motion is assured by symmetry in the model systems as well as in the exact dynamics. Let us denote the mean velocity field by $\bar{u}_i(\mathbf{x}, t)$, when it does not vanish, and continue to denote the fluctuating field, with zero ensemble mean, by $u_i(\mathbf{x}, t)$. Then the Navier–Stokes equation in x space breaks up into the two equations

$$(\partial/\partial t - \nu\nabla^2)\,\bar{\mathbf{u}} + \bar{\mathbf{u}}\,.\,\nabla\bar{\mathbf{u}} = -\langle\mathbf{u}\,.\,\nabla\mathbf{u}\rangle = \nabla\bar{p}, \tag{5.1}$$

$$(\partial/\partial t - \nu\nabla^2)\,\mathbf{u} + \bar{\mathbf{u}}\,.\nabla\mathbf{u} + \mathbf{u}\,.\nabla\bar{\mathbf{u}} = -(\mathbf{u}\,.\nabla\mathbf{u} - \langle\mathbf{u}\,.\nabla\mathbf{u}\rangle) - \nabla p, \tag{5.2}$$

where $\bar{p}$ and p are the mean and fluctuating pressure. In our abstract notation, (5.1) translates to

$$d\bar{u}_\alpha(t)/dt + \nu_{\alpha\beta}\bar{u}_\beta(t) = A_{-\alpha\beta\gamma}[\bar{u}_\beta(t)\,\bar{u}_\gamma(t) + U_{\beta,-\gamma}(t)]. \tag{5.3}$$

Equation (5.3) is retained without approximation in both direct-interaction and test-field models.

Equation (5.2) translates to

$$du_\alpha/dt + \nu_{\alpha\beta}u_\beta - \bar{A}_{-\alpha\beta\gamma}u_\beta u_\gamma = A_{-\alpha\beta\gamma}(u_\beta u_\gamma - U_{\beta,-\gamma}), \tag{5.4}$$

which replaces (3.1). It is not really clear how to include properly the effects of mean velocity in the test-field equations of motion (3.2) and (3.3). Since these equations are linear in the test field, and only the response matrices are of eventual interest, a mean part to the test field itself is pointless. The rationale of the test-field equations is use of the coupling between $\mathbf{v}^S$ and $\mathbf{v}^C$ as a measure of that self-distortion of the turbulence which limits the memory times for energy transfer. On this basis, one can argue both for and against including terms $\bar{\mathbf{u}}\,.\,\nabla\mathbf{v}^C$ and $\bar{\mathbf{u}}\,.\,\nabla\mathbf{v}^S$ in the equations. The shear associated with the mean velocity field certainly distorts the flow but, on the other hand, it does so coherently. We shall elect to omit mean-field terms entirely from (3.2) and (3.3) on the basis of a practical consideration. Our assurance that the square-root matrices h and h^G could be formed properly depended on the fact that G^S and G^C are Hermitian matrices. The operator $\bar{\mathbf{u}}\,.\,\nabla$ is anti-Hermitian, and inclusion of mean-field terms would therefore destroy the hermiticity of G^S and G^C.

Equation (5.4) implies that the test-field model and direct-interaction amplitude equations (3.10) and (4.1) should be altered to, respectively,

$$du_\alpha(t) - \bar{A}_{-\alpha\beta\gamma}u_\beta(t)\,\bar{u}_\gamma(t) + \eta_{\alpha\beta}(t)\,u_\beta(t) = w(t)\,C_{-\alpha\beta\gamma}(t)\,[\xi_\beta(t)\,\xi_\gamma(t) - U_{\beta,-\gamma}(t)], \tag{5.5}$$

$$du_\alpha(t) - \bar{A}_{-\alpha\beta\gamma}u_\beta(t)\,\bar{u}_\gamma(t) + \int_0^t \eta_{\alpha\beta}(t,s)\,u_\beta(s)\,ds = A_{-\alpha\beta\gamma}[\xi_\beta(t)\,\xi_\gamma(t) - U_{\beta,-\gamma}(t)]. \tag{5.6}$$

In accord with the preceeding discussion, there are no changes in (3.11) or (3.12). The resultant changes in the final statistical equations consist of addition of terms linear in $\bar{\mathbf{u}}$ to the left-hand sides of (3.16) and (4.3)–(4.6). These additions are given in table 1.

Equation	Addition to left-hand side
(3.16)	$-\bar{A}_{-\alpha\mu\gamma} U_{\mu\beta}(t)\, \bar{u}_\gamma(t) - \bar{A}^*_{-\beta\mu\gamma} U_{\alpha\mu}(t) \bar{u}^*_\gamma(t)$
(4.3)	$-\bar{A}_{-\alpha\mu\gamma} U_{\mu\beta}(t) \bar{u}_\gamma(t) - \bar{A}^*_{-\beta\mu\gamma} U_{\alpha\mu}(t) \bar{u}^*_\gamma(t)$
(4.4)	$-\bar{A}_{-\alpha\mu\gamma} U_{\mu\beta}(t,t') \bar{u}_\gamma(t)$
(4.5)	$-\bar{A}_{-\alpha\mu\gamma} G_{\mu\beta}(t,t') \bar{u}_\gamma(t)$
(4.6)	$-\bar{A}_{-\alpha\mu\gamma} U_{\mu\beta}(t,t') \bar{u}_\gamma(t)$

TABLE 1

The changes in these five statistical equations all arise from the additions on the left-hand sides of (5.5) and (5.6), and are the same for corresponding test-field and direct-interaction equations. The $U_{\beta,-\gamma}(t)$ terms on the right-hand sides of (5.5) and (5.6) lead to no explicit changes in (3.16) and (4.3)–(4.6) because they cancel contributions involving $\langle \xi_\beta(t)\, \xi_\gamma(t) \rangle$, which implicitly were assumed to vanish in the analysis of §§ 2–4. Equation (5.3) completes the statistical equations, for both the test-field and direct-interaction cases.

The boundary conditions assumed up to now are cyclic on a very large box. The box was taken large only in order to permit exact isotropy, so that this restriction is not required in the present matrix equations. The much more general boundary condition of zero velocity on the boundary of a domain of arbitrary size and shape, and hybrid conditions where there is slip (zero stress) or cyclic behaviour over part of the boundary, can be handled by the techniques previously introduced (in x space) for the direct-interaction approximation. Let V be a closed (possibly multiply-connected) volume with boundary surface B. If all velocity components vanish on B, then the pressure can be eliminated to leave the x space Navier–Stokes equation in the form (Kraichnan 1964*b*)

$$(\partial/\partial t - \nu\nabla^2)\, u_i + L_{ij}(\nabla)\, u_j = -P_{ij}(\nabla)\, [\partial/\partial x_m(u_j u_m)]. \tag{5.7}$$

Here $P_{ij}(\nabla)$ and $L_{ij}(\nabla)$ are defined by†

$$P_{ij}(\nabla)\, [f(\mathbf{x})] = \delta_{ij} f(\mathbf{x}) - (\partial/\partial x_i) \int_V D(\mathbf{x},\mathbf{y})\, \partial f(\mathbf{y})/\partial y_j\, d^3y, \tag{5.8}$$

$$L_{ij}(\nabla)\, [f(\mathbf{x})] = \nu(\partial/\partial x_i) \int_B D(\mathbf{x},\mathbf{y})\, n_j(\mathbf{y})\, n_r(\mathbf{y})\, n_s(\mathbf{y})\, (\partial^2 f(\mathbf{y})/\partial y_r \partial y_s)\, d^2y, \tag{5.9}$$

where $f(\mathbf{x})$ is a test function which vanishes on B, $\mathbf{n}(\mathbf{y})$ is the inward-pointing normal unit vector to $B(\mathbf{y})$, and $D(\mathbf{x},\mathbf{y})$ is the Green's function whose normal derivative vanishes on B and which satisfies

$$\nabla^2_x D(\mathbf{x}, y) = \delta^3(\mathbf{x}-\mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \text{ in } V). \tag{5.10}$$

The simplest translation into our matrix notation is obtained if we expand the velocity field in the complete set of orthonormal eigenfunctions $\psi_n(\mathbf{x})$ which vanish on B and satisfy

$$\nabla^2 \psi_n(\mathbf{x}) = \lambda_n \psi_n(\mathbf{x}). \tag{5.11}$$

Thus,

$$u_i(\mathbf{x},t) = \Sigma_n\, u_{i,n}(t)\, \psi_n(\mathbf{x}), \tag{5.12}$$

† Equation (5.8) corrects an error in equation (2.10) of Kraichnan (1964*b*).

and we take the $u_{i,n}(t)$ as the variables $u_\alpha(t)$. With the identification

$$\alpha \to (i,n), \quad \beta \to (j,r), \quad \gamma \to (m,s),$$

we have, setting $\Pi_{ij}(\nabla) = \delta_{ij} - P_{ij}(\nabla)$,

$$\nu_{\alpha\beta} = -\delta_{\alpha\beta}\lambda_n \quad \text{(note that } \lambda_n < 0, \text{ all } n), \tag{5.13}$$

$$L_{\alpha\beta} = \int \psi_n(\mathbf{x})\, L_{ij}(\nabla)\, \psi_r(\mathbf{x})\, d^3x, \tag{5.14}$$

$$A_{\alpha\beta\gamma} = -\int \psi_n(\mathbf{x})\, P_{ia}(\nabla)\, (\partial/\partial x_b)\, \{[P_{aj}(\nabla)\, \psi_r(\mathbf{x})]\, [P_{bm}(\nabla)\, \psi_s(\mathbf{x})]\}\, d^3x, \tag{5.15}$$

$$B_{\alpha\beta\gamma} = -\int \psi_n(\mathbf{x})\, \Pi_{ia}(\nabla)\, (\partial/\partial x_b)\, \{[P_{aj}(\nabla)\, \psi_r(\mathbf{x})]\, [P_{bm}(\nabla)\, \psi_s(\mathbf{x})]\}\, d^3x. \tag{5.16}$$

Here we note that $\int \psi_n(\mathbf{x})\, \psi_r(\mathbf{x})\, d^3x = \delta_{nr}$, so that the kinetic energy is a sum of squares. The $\psi_n(\mathbf{x})$ are real; therefore, we make the identification $-\alpha \to \alpha$ and sum over positive α values only in all matrix operations.

The only changes in the equations of §§3 and 4 due to the change of boundary conditions involve $L_{\alpha\beta}$. In (3.1), (3.13), (4.2), (5.3) and (5.4), $\nu_{\alpha\beta}$ is replaced by $\nu_{\alpha\beta} + L_{\alpha\beta}$. There is no change in (3.2), (3.3), (3.14) and (3.15) because the equations for $\mathbf{v}^S$ and $\mathbf{v}^C$ do not involve the pressure, from which $L_{\alpha\beta}$ arises.

Several additional modifications can now be made. First of all, there are no changes in the equations if the tangential velocity on the boundary has prescribed time-independent solenoidal values instead of being zero. It *is* necessary to modify the basis-function set $\psi_n(x)$. This can be done in general as follows. We set up three mutually perpendicular vector fields in V such that, on B, there are everywhere two tangential and one normal field. Then, for each of the three fields, we construct an orthonormal set of eigenfunctions obeying (5.11) with the correct boundary conditions. For example, suppose that V is a rectangular box with one pair of opposite walls sliding in opposite directions (bounded Couette flow). Then the three vector fields are ordinary Cartesian unit vectors and the eigenfunctions for each field are appropriate sine and cosine combinations.†

A similar change in basis functions handles the case where there is zero tangential stress (zero normal derivative of tangential velocity) on part of the boundary. In this case there is, additionally, the change (Kraichnan *1964b*) that the integration in (5.9) omits the zero-stress part of B.

Finally, part of the boundary can be at infinity. In the case of an infinitely ong straight channel of constant, arbitrary cross-section, it is convenient to take cyclic boundary conditions, with large period, in the axial direction and expand in products of wavenumber modes, in the axial direction, using eigenfunctions of the two-dimensional Laplacian in the cross-sectional planes. Here an external pressure gradient can be represented by an additional term $-\nabla\bar{p}_{\text{ext}}$ on the right-hand side of (5.1), where $\bar{p}_{\text{ext}}$ obeys Laplace's equation in V and has zero normal derivative on all finite parts of B. This adds to the right-hand side of (5.3) the term

$$\bar{f}_\alpha(t) = -\int \psi_n^*(\mathbf{x})\, (\partial/\partial x_i)\, \bar{p}_{\text{ext}}(\mathbf{x},t)\, d^3x, \tag{5.17}$$

† If the geometry is curved, or for other reasons the three vector fields change direction as functions of position, appropriate modification must be made in (5.14)–(5.16) so that the spatial derivatives of the velocity field are correctly taken.

where $\psi_n(x)$ is the three-dimensional basis function for mode α, as shown in (5.12). The u_α are now complex again, so that sums run over both positive and negative α values.

For any of the various boundary conditions and geometries, choices for the u_α other than those indicated above may prove more advantageous for computation. For example, the Gibbs phenomenon, associated with the fact that the normal velocity and its normal derivative both vanish on B wherever the tangential velocity is prescribed to be solenoidal, is reduced in severity if we adopt, for this velocity component, the Chandrasekhar–Reid functions, which are eigenfunctions of the squared Laplacian (Chandrasekhar & Reid 1957). Alternatively, convergence of the modal expansions can be enhanced by suitable choices of orthogonal polynomials as basis functions (Orszag 1971). A precaution to be noted here is that the transformation theory of § 3 must be appropriately generalized if the weight function of the polynomials is not a constant, for, in that case, the kinetic energy is no longer a simple sum of squares of mode amplitudes.

6. Discussion

The preceding analysis has brought the equations of the test-field model and the direct-interaction approximation into a uniform notation and to a stage where they can be fairly directly programmed for computation of turbulence in interesting geometries like channels and boxes. We wish now to compare the two approximations with regard to probable faithfulness of representation of the turbulence and difficulty of computation.

We have noted before that the white-noise forcing term fundamental to the test-field model is foreign to actual turbulence dynamics because there is, in the latter, no clean division into short and long time scales. Moreover the test-field model is more arbitrary, within its general theoretical framework, than the direct-interaction approximation, which represents an essentially unique modelling of the actual dynamics if certain basic invariances and symmetries of the latter are retained. In I it was noted that the interaction of the test-field components $\mathbf{v}^S$ and $\mathbf{v}^C$ is at best only a measure of the sought-after memory times for energetic interaction; the theory remains as plausible if the coefficients $B_{\alpha\beta\gamma}$ in (3.1), (3.2) and the equations which follow are scaled by a parameter g to alter the effective strength of coupling of $\mathbf{v}^S$ and $\mathbf{v}^C$. Comparison of the integrated test-field equations with isotropic-turbulence experiments suggests that a g in the range 1·0 to 1·5 is the best (Herring & Kraichnan 1972). The white-noise aspect of the test-field model is sufficiently artificial that the model should probably be used only to determine the single-time correlations $U_{\alpha\beta}(t)$. The direct-interaction approximation, on the other hand, gives qualitatively correct difference-time behaviour for $U_{\alpha\beta}(t,t')$ and, so far, fairly good quantitative agreement with computer experiments on two-time correlations in decaying isotropic turbulence (Orszag & Patterson 1972).

The one qualitative theoretical superiority of the test-field model over the direct-interaction approximation is its stochastic Galilean invariance (see I), which makes it yield a $-\frac{5}{3}$ inertial range spectrum, which is much closer to observation than the $-\frac{3}{2}$ spectrum of the direct-interaction approximation. The latter seems clearly to be an artifact due to violation of Galilean invariance, whatever the true inertial-range law may turn out to be. This difference between the two approximations probably is not of great significance in problems, like channel flow, where the overall transport properties and spectral distribution in the energy-bearing wavenumbers is much more important than the precise shape of the spectral tail at high wavenumbers. We should note that the abridged Lagrangian-history direct-interaction approximation, which is Galilean-invariant (Kraichnan 1966), can also be formulated for inhomogeneous problems in the form of §§4 and 5. For a given problem, it requires roughly three times as much programming and computation as the direct-interaction approximation.

The principal advantage to be sought from the test-field model is shortened machine computation times over those for the direct-interaction approximation. Let us now examine this point in some detail. Suppose that N modes are retained in a calculation which proceeds for n time steps. If all the matrices are diagonal, the number of arithmetic operations required for the direct-interaction equations is of order N^3n^3, where the n^3 comes from the need to compute two-time quantities and the integrals over past times. This estimate assumes that there are no zeros in the A coefficients. The corresponding test-field model figure is N^3n, since only the current time is involved at each time step. However, if the number of non-zero A's is of $O(N^2)$, rather than $O(N^3)$, as in fact it is in the Fourier representation, then the figure for the direct-interaction equations is of order N^2n^3, while the test-field estimate remains N^3n, because of the θ equations, (2.24).

In the general non-diagonal case it can be seen from the analysis of §§3–5 that the number of operations for the direct-interaction equations grows to N^7n^3, with no zeros in the A's or to N^5n^3, if the number of non-zero A's is of $O(N^2)$. In the test-field equations, the operations whose number increases as the highest powers of N are the integration of the θ equations and the diagonalization of the θ functions at each step, so as to permit computation of h and h^G. Integration of the θ equations takes $O(N^7n)$ operations, and there is no reduction if the A's and B's contain zeros. Diagonalization of $\theta_{\alpha\beta\gamma\mu\epsilon\lambda}(t)$ can be carried out by treating it as a two-dimensional square matrix (arranging $\alpha\beta\gamma$ and $\mu\epsilon\lambda$ as linear arrays) and then applying standard techniques. Handled in this way, θ and θ^G are $N^3 \times N^3$ matrices, so that a general diagonalization method would require $O(N^9)$ operations per time step.

In contrast to the preceeding estimates, direct integration of the Navier–Stokes equation in the form (5.4) takes $O(N^3n)$ operations in general, and $O(N^2n)$ operations if there are only $O(N^2)$ non-zero A's. When the geometry permits, fast-Fourier-transform techniques can reduce this estimate to $O(N \ln (N)\, n)$ (Patterson & Orszag 1971). The computational advantage of closure approximations over direct integration comes from the fact that a direct integration must treat each mode individually, while the statistical quantities which enter the closure equations are smooth functions of time and mode-index arguments, and hence can be well-described by parametrizations (e.g., expansions in well-chosen orthogonal functions) with relatively few numbers. In other words, apt representations of the functions U, η, etc., can replace N and n by smaller values

N_{eff} and n_{eff} in the estimates for the direct-interaction and test-field models. We are referring here not to the choice of orthogonal co-ordinates u_α, but to expansions of statistical quantities in functions of given u_α. The integration of the direct-interaction equations for isotropic turbulence (Kraichnan 1964*a*) provides an illustration. Here 20 parameters yield an adequate description of the covariance matrix (for given time arguments) over a wavenumber range of 30 to 1, a situation in which there are over 10^4 degrees of freedom. Thus N/N_{eff} is the order of 10^3. Less spectacular ratios are to be expected when the spatial symmetry is less restrictive.

In the diagonal case, the test-field model appears to have a definite computational advantage over the direct-interaction approximation. However, in the non-diagonal case, the $O(N^9n)$ operations for diagonalization of θ would appear to overwhelmingly destroy this advantage. We have the ironical situation that the hoped-for computational simplicity of the test-field model, which was its principal motivation, is thwarted, in the general case, because of problems involving the most arbitrary part of the model, namely the memory integrals. In the next section a simplifying alteration of the model is proposed which restores the clear computational advantage over direct-interaction calculations. The altered model, which involves negligible quantitative changes, can be regarded as a generalization of Orszag's eddy-damped Markovian model (Orszag 1974; Leith 1971).

7. Simplifying modification

Let us make the definition

$$\eta^G_{\alpha\beta}(t) = \eta^S_{\alpha\beta}(t) + \eta^C_{\alpha\beta}(t),$$

thereby combining the (non-mixing) transverse and longitudinal parts of the test-field damping matrix. We may then define the characteristic times

$$\theta_{\alpha\beta}(t) = \int_0^t G'_{\alpha\beta}(t,s)\,ds, \tag{7.1}$$

where

$$dG'_{\alpha\beta}(t,t')/dt + \tfrac{3}{2}[\eta^G_{\alpha\gamma}(t)\,G'_{\gamma\beta}(t,t') + G'_{\alpha\gamma}(t,t')\,\eta^G_{\gamma\beta}(t)] = 0, \quad G'_{\alpha\beta}(t',t') = \delta_{\alpha\beta}. \tag{7.2}$$

Then we have

$$d\theta_{\alpha\beta}(t)/dt + \tfrac{3}{2}[\eta^G_{\alpha\gamma}(t)\,\theta_{\gamma\beta}(t) + \theta_{\alpha\gamma}(t)\,\eta^G_{\gamma\beta}(t)] = \delta_{\alpha\beta}, \quad \theta_{\alpha\beta}(0) = 0. \tag{7.3}$$

The analysis of §3 then shows that $\theta_{\alpha\beta}(t)$ is a Hermitian matrix with positive eigenvalues. Let $D_{\alpha\beta}(t)$ be the unitary matrix which diagonalizes $\theta_{\alpha\beta}(t)$. Then,

$$\theta_{\alpha\beta}(t) = D^*_{\gamma\alpha}(t)\,D_{\gamma\beta}(t)\,\theta_\gamma(t), \tag{7.4}$$

where the $\theta_\gamma(t)$ are these eigenvalues. Now define the matrix $\theta'_{\alpha\beta\gamma\mu\epsilon\lambda}(t)$ as follows. The eigenvalues of θ', denoted by $\theta'_{\alpha\beta\gamma}(t)$, satisfy

$$3[\theta'_{\alpha\beta\gamma}(t)]^{-1} = [\theta_\alpha(t)]^{-1} + [\theta_\beta(t)]^{-1} + [\theta_\gamma(t)]^{-1}, \tag{7.5}$$

and θ' is given in terms of its eigenvalues by

$$\theta'_{\alpha\beta\gamma\mu\epsilon\lambda} = D^*_{\tau\alpha}D^*_{\nu\beta}D^*_{\pi\gamma}D_{\tau\mu}D_{\nu\epsilon}D_{\pi\lambda}\theta'_{\tau\nu\pi}. \tag{7.6}$$

It follows that the matrix θ' transforms under general unitary transformation of the u_α according to the rule following (3.7).

Our modification of the test-field model consists now of replacing $\theta_{\alpha\beta\gamma\mu\epsilon\lambda}(t)$ and $\theta^G_{\alpha\beta\gamma\mu\epsilon\lambda}(t)$, which were defined by (3.6), with $\theta'_{\alpha\beta\gamma\mu\epsilon\lambda}(t)$. That is, we replace $\theta_{\alpha\beta\gamma\mu\epsilon\lambda}(t)$ by the components of θ' all of whose indices denote a solenoidal mode, and replace $\theta^G_{\alpha\beta\gamma\mu\epsilon\lambda}(t)$ by those components of θ' whose indices are all solenoidal except the first and fourth, which are longitudinal. To see the implications of the modification, let us return to isotropic turbulence, where all matrices are diagonal. In this case, our modification replaces $\theta_{\alpha\beta\gamma}(t)$ and $\theta^G_{\alpha\beta\gamma}(t)$, of §2, by $\theta'_{\alpha\beta\gamma}(t)$. It may immediately be verified that $\theta_{\alpha\beta\gamma}(t)$ and $\theta^G_{\alpha\beta\gamma}(t)$ are identical in value with the corresponding components of $\theta'_{\alpha\beta\gamma}(t)$ both at very small t, when

$$\theta'_{\alpha\beta\gamma}(t) \approx t,$$

and in the statistically steady state, where (to take the all-solenoidal case)

$$\theta_{\alpha\beta\gamma}(\infty) = \theta'_{\alpha\beta\gamma}(\infty) = [\eta^S_{\alpha\alpha}(\infty) + \eta^S_{\beta\beta}(\infty) + \eta^S_{\gamma\gamma}(\infty)]^{-1} \quad \text{(not summed)}. \tag{7.7}$$

Moreover, there is identity of value for all t if α, β and γ refer to modes whose wave vectors form an equilateral triangle.

Thus, the modification affects the results for isotropic turbulence only in the middle transient period. Here, some numerical examples with three-mode systems suggest that the changes in memory-integral values are typically the order of 1 %. This is supported by repetitions of isotropic turbulence decay calculations (Herring & Kraichnan 1972), which show negligible changes as a result of the modification. The modified test-field model thus seems as plausible an approximation for extension to inhomogeneous turbulence as the original model. The key equation in the modification, equation (7.5), was suggested by, and can be considered a generalization of the eddy-damped Markovian model introduced by Orszag (Orszag 1974; Leith 1971).

In effect, (7.5) and (7.3) serve to determine the reciprocal characteristic time for the triad $(\alpha\beta\gamma)$ as an average over reciprocal times for the equilateral-triangle triads $(\alpha\alpha\alpha)$, $(\beta\beta\beta)$ and $(\gamma\gamma\gamma)$.

In order to realize computing economies from the modified test-field model, the effective interaction coefficients $C_{\alpha\beta\gamma}(t)$ and $D_{\alpha\beta\gamma}(t)$, defined by (3.9), are evaluated in the diagonal representation. Then they may be transformed back to whatever representation is in use, or the integrations at each time step can be carried out in the (continually changing) diagonal representation, and results transformed back at the end. The latter procedure probably is slightly more economical, but we shall describe only the former, because it is more straightforward. The 'bare' coefficients in the diagonal representation are

$$A'_{\alpha\beta\gamma}(t) = D^*_{\alpha\mu}(t)\, D^*_{\beta\epsilon}(t)\, D^*_{\gamma\lambda}(t)\, A_{\mu\epsilon\lambda}, \quad B'_{\alpha\beta\gamma}(t) = D^*_{\alpha\mu}(t)\, D^*_{\beta\epsilon}(t)\, D^*_{\gamma\lambda}(t)\, B_{\mu\epsilon\lambda}, \tag{7.8}$$

while the back transformation yields

$$\begin{aligned} C_{\alpha\beta\gamma}(t) &= D_{\mu\alpha}(t)\, D_{\epsilon\beta}(t)\, D_{\lambda\gamma}(t)\, A'_{\mu\epsilon\lambda}(t)\, [\theta'_{\mu\epsilon\lambda}(t)]^{\frac{1}{2}}, \\ D_{\alpha\beta\gamma}(t) &= D_{\mu\alpha}(t)\, D_{\epsilon\beta}(t)\, D_{\lambda\gamma}(t)\, B'_{\mu\epsilon\lambda}(t)\, [\theta'_{\mu\epsilon\lambda}(t)]^{\frac{1}{2}}, \end{aligned} \tag{7.9}$$

where the positive square roots are taken. Here the sorting out of solenoidal and longitudinal indices, associated with h and h^G, occurs automatically, as a consequence of past definitions.

Our complete set of equations is now (3.13)–(3.16), (5.3), (7.3)–(7.5), (7.8) and (7.9), together with associated definitions, and with the mean-field and boundary terms as tabulated in § 5. The principal computational difference from the previous set is that now we have only to diagonalize the $N \times N$ matrix $\theta_{\alpha\beta}(t)$, instead of an $N^3 \times N^3$ matrix. Determination of $D_{\alpha\beta}(t)$ at all the time steps requires $O(N^3 n)$ operations, returning to the discussion in § 6, while (7.8) and (7.9) require $O(N^6 n)$ operations. Thus the computational economy of the test-field model over the direct-interaction approximation now carries over undiminished to the general inhomogeneous case.

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